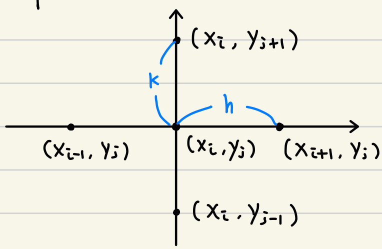
05/21/20

Sec 12.1. Elliptic Partial Differential Equations

$-\Delta u + \vec{\beta}\cdot\nabla u + \gamma u = f$

$$-\Delta u = f \quad \text{in } \Omega \qquad \text{(Poisson)}$$
$$u = g \quad \text{on } \partial\Omega \qquad \text{(Dirichlet)}$$

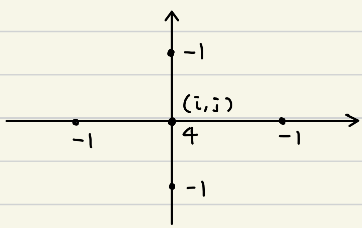
- Grid points

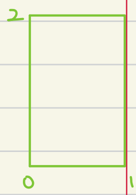


- FDM

$$\Delta u = u_{xx} + u_{yy}$$
$$\approx \frac{w_{i+1,j} - 2w_{i,j} + w_{i-1,j}}{h^2} + \frac{w_{i,j+1} - 2w_{i,j} + w_{i,j-1}}{k^2}$$
$$= \frac{1}{h^2}(w_{i+1,j} + w_{i-1,j}) + \frac{1}{k^2}(w_{i,j+1} + w_{i,j-1})$$
$$- \left(\frac{2}{h^2} + \frac{2}{k^2}\right) w_{i,j}$$

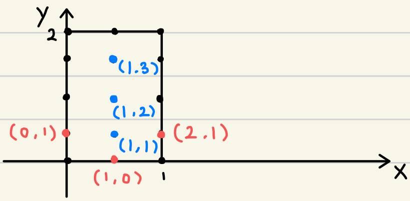
If $h = k$





ex) $\Delta u = 4, \quad 0 < x < 1, \quad 0 < y < 2$

$$u(x,0) = x^2, \quad u(x,2) = (x-2)^2, \quad 0 \le x \le 1$$
$$u(0,y) = y^2, \quad u(1,y) = (y-1)^2, \quad 0 \le y \le 2$$
$$h = k = 1/2.$$



$u(0,h) = h^2$

$$(1,1): \quad \frac{W_{0,1} - 2W_{1,1} + W_{2,1}}{h^2} + \frac{W_{1,0} - 2W_{1,1} + W_{1,2}}{h^2} = 4$$

$$(1,2): \quad \frac{W_{0,2} - 2W_{1,2} + W_{2,2}}{h^2} + \frac{W_{1,1} - 2W_{1,2} + W_{1,3}}{h^2} = 4$$

$$(1,3): \quad \frac{W_{0,3} - 2W_{1,3} + W_{2,3}}{h^2} + \frac{W_{1,2} - 2W_{1,3} + W_{1,4}}{h^2} = 4$$

$$\Rightarrow (1,1): \quad -4W_{1,1} + W_{1,2} = 4h^2 - h^2 - (h-1)^2 - h^2$$

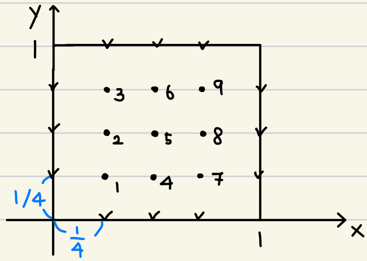
$$(1,2): \quad W_{1,1} - 4W_{1,2} + W_{1,3} = 4h^2 - (2h)^2 - (2h-1)^2$$

$$(1,3): \quad W_{1,2} - 4W_{1,3} = 4h^2 - (3h)^2 - (3h-1)^2 - (h-2)^2$$

$$\Rightarrow \begin{bmatrix} -4 & 1 & 0 \\ 1 & -4 & 1 \\ 0 & 1 & -4 \end{bmatrix} \begin{bmatrix} W_{1,1} \\ W_{1,2} \\ W_{1,3} \end{bmatrix} = \vec{b}(f, h)$$

$$A\vec{w} = \vec{b} \quad (\vec{w} = A^{-1}\vec{b})$$

ex)



$h = k = 0.25$

$\Delta u \Rightarrow A\vec{w}$ (9 x 9 matrix)

$$\begin{bmatrix} 4 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 4 & -1 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 4 & -1 & 0 & -1 & 0 & 0 & 0 \\ -1 & 0 & -1 & 4 & -1 & 0 & -1 & 0 & 0 \\ 0 & -1 & 0 & -1 & 4 & -1 & 0 & -1 & 0 \\ 0 & 0 & -1 & 0 & -1 & 4 & -1 & 0 & -1 \\ 0 & 0 & 0 & -1 & 0 & -1 & 4 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & -1 & 4 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & -1 & 4 \end{bmatrix}$$

$\Rightarrow$ A : symmetric positive definite (SPD)

Def) positive definite $A \in \mathbb{R}^{N \times N}$ if
$\vec{x}^T A \vec{x} > 0$ for all $\vec{x} \in \mathbb{R}^N$

Lemma) SPD $\Rightarrow$ invertible

Quiz 5.

1. Consider the following boundary-value problem.

$$y'' = -e^{-2y}, \quad 1 \le x \le 2,$$
$$y(1) = 0, \quad y(2) = \ln 2, \quad h = 1/3$$

Apply the finite-difference method to get the 2x2 nonlinear system. Find the first Newton's update from the initial guess (0,0).

2. For a given interval [x_i,x_{i+1}], find the three piecewise-quadratic basis functions.