05/26/20

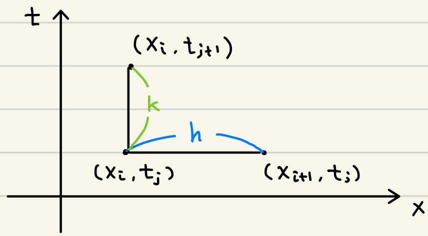
Sec 12.2. Parabolic Partial Differential Equations

$$\frac{\partial u}{\partial t}(x,t) = \alpha^2 \frac{\partial^2 u}{\partial x^2}(x,t),$$

$$0 < x < \ell, \quad t > 0.$$

BC : $u(0,t) = u(\ell,t) = 0, \quad t > 0$

IC : $u(x,0) = f(x), \quad 0 \le x \le \ell.$



Let $v_i(t)$ be an approximation for $u(x_i, t)$ when FDM in x-direction is applied.

$$\frac{\partial^2 u}{\partial x^2}(x_i, t) \approx \frac{v_{i+1}(t) - 2v_i(t) + v_{i-1}(t)}{h^2}$$

$$\frac{\partial u}{\partial t}(x_i, t) \approx \frac{dv_i}{dt}(t)$$

$$\Rightarrow \quad \frac{dv_i}{dt}(t) = \alpha^2 \frac{v_{i+1}(t) - 2v_i(t) + v_{i-1}(t)}{h^2}$$

↳ System of DEs (IVPs)

BC $\longrightarrow$ $v_0(t) = v_N(t) = 0$

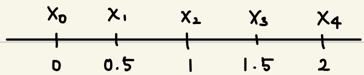
IC $\longrightarrow$ $v_i(0) = f(x_i)$

ex) $u_t = u_{xx}, \quad 0 < x < 2, \quad t > 0$

BC : $u(0,t) = u(2,t) = 0$

IC : $u(x,0) = \sin(\pi x/2), \quad 0 \le x \le 2$

' $h = 0.5$ '



$$\frac{dv_1}{dt}(t) = \frac{v_2(t) - 2v_1(t) + \overset{0}{\overset{\|}{v_0(t)}}}{h^2}$$

$$\frac{dv_2}{dt}(t) = \frac{v_3(t) - 2v_2(t) + v_1(t)}{h^2}$$

$$\frac{dv_3}{dt}(t) = \frac{\overset{0}{\overset{\|}{v_4(t)}} - 2v_3(t) + v_2(t)}{h^2}$$

↳ System of DEs (IVP)

IC : $v_1(0) = f(x_1) = \sin(\pi(0.5)/2)$
$= \sqrt{2}/2$

$v_2(0) = f(x_2) = \sin(\pi/2) = 1$

$v_3(0) = f(x_3) = \sin(\pi(1.5)/2)$
$= \sqrt{2}/2$

$$\frac{d\vec{v}}{dt}(t) = \vec{f}(t, \vec{v}(t))$$

$$\vec{v}(0) = [\sqrt{2}/2 \quad 1 \quad \sqrt{2}/2]^T$$

↳ Forward, Backward, C-N, RK4

Let $\vec{w}_j$ be an approximation for $\vec{v}(t_j)$ when numerical methods in t-direction are applied.

- Forward Difference Method

$$\frac{\vec{w}_{j+1} - \vec{w}_j}{k} = \vec{f}(t_j, \vec{w}_j),$$
$$\vec{w}_0 = \vec{v}(0)$$

- $\vec{f}(t_j, \vec{w}_j) = \vec{f}(\vec{w}_j)$ ← a linear map

$$= \frac{1}{h^2}\begin{bmatrix} -2 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -2 \end{bmatrix}\begin{bmatrix} w_{1,j} \\ w_{2,j} \\ w_{3,j} \end{bmatrix}$$

$$= \frac{1}{h^2} B \vec{w}_j$$

$$\Rightarrow \vec{w}_{j+1} = \vec{w}_j + \frac{k}{h^2} B \vec{w}_j$$

- Backward Method

$$\frac{\vec{w}_{j+1} - \vec{w}_j}{k} = \vec{f}(t_{j+1}, \vec{w}_{j+1}),$$
$$\vec{w}_0 = \vec{v}(0).$$

$$\Rightarrow \left(I - \frac{k}{h^2} B\right) \vec{w}_{j+1} = \vec{w}_j$$

$$\Rightarrow \vec{w}_{j+1} = \left(I - \frac{k}{h^2} B\right)^{-1} \vec{w}_j$$

- Crank - Nicolson Method

$$\frac{\vec{w}_{j+1} - \vec{w}_j}{k} = \frac{1}{2}\left[\vec{f}(t_j, \vec{w}_j) + \vec{f}(t_{j+1}, \vec{w}_{j+1}) \right]$$

$$\Rightarrow \vec{w}_{j+1} - \vec{w}_j = \frac{k}{2h^2}\left(B\vec{w}_j + B\vec{w}_{j+1} \right)$$

$$\Rightarrow \left(I - \frac{k}{2h^2}B \right)\vec{w}_{j+1} = \left(I + \frac{k}{2h^2}B \right)\vec{w}_j$$

$$\Rightarrow \vec{w}_{j+1} = \left(I - \frac{k}{2h^2}B \right)^{-1}\left(I + \frac{k}{2h^2}B \right)\vec{w}_j$$

- RK 4 .