05/28/20

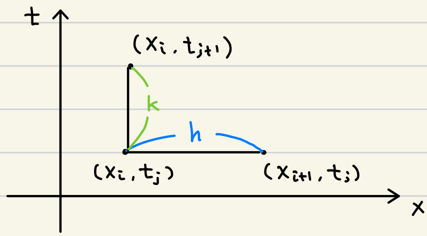
Sec 12.3. Hyperbolic Partial Differential Equations.

$$\frac{\partial^2 u}{\partial t^2} - \alpha^2 \frac{\partial^2 u}{\partial x^2} = 0, \quad 0 < x < l, \quad t > 0$$

BC : $u(0,t) = u(l,t) = 0, \quad t > 0$

IC : $u(x,0) = f(x), \quad (\partial u/\partial t)(x,0) = g(x),$

$$0 \le x \le l.$$



$$\frac{\partial^2 u}{\partial t^2}(x_i, t_j) \approx \frac{W_{i,j+1} - 2W_{i,j} + W_{i,j-1}}{k^2}$$

$$\frac{\partial^2 u}{\partial x^2}(x_i, t_j) \approx \frac{W_{i+1,j} - 2W_{i,j} + W_{i-1,j}}{h^2}$$

$$\Rightarrow \frac{W_{i,j+1} - 2W_{i,j} + W_{i,j-1}}{k^2} - \alpha^2 \frac{W_{i+1,j} - 2W_{i,j} + W_{i-1,j}}{h^2} = 0$$

BC $\longrightarrow$ $W_{0,j} = W_{N,j} = 0$

IC $\longrightarrow$ $W_{i,0} = f(x_i)$

$(W_{i,1} - W_{i,0})/k = g(x_i)$

ex) $u_{tt} - u_{xx} = 0, \quad 0 < x < 1, \quad t > 0$

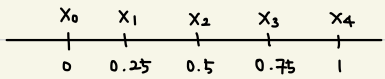
BC : $u(0,t) = u(1,t) = 0, \quad t > 0$

IC : $u(x,0) = \sin \pi x, \quad u_t(x,0) = 0.$

$0 \le x \le 1.$

$h = k = 0.25$

- x - direction



- t - direction

$t = 0.5$ (new update from PDEs)

$t = 0.25$ (IC 2 : $g(x)$)

$t = 0$ (IC 1 : $f(x)$)

(1) $t = 0.25, \quad x = 0.25$

$u(x_1, 0) = f(x_1) = \sin \pi x_1$ (IC)

0 (BC)

$$\frac{W_{1,2} - 2W_{1,1} + W_{1,0}}{k^2} - \frac{W_{2,1} - 2W_{1,1} + W_{0,1}}{h^2} = 0$$

$$\hookrightarrow \quad W_{i,1} = W_{i,0} + k\,g(x_i)$$
$$= f(x_i) + k\,g(x_i)$$
$$= \sin \pi x_i, \qquad 1 \le i \le 3$$

(2) $t = 0.25, \quad x = 0.5$

$$\frac{W_{2,2} - 2W_{2,1} + \overset{\sin \pi x_2}{W_{2,0}}}{k^2} - \frac{W_{3,1} - 2W_{2,1} + W_{1,1}}{h^2} = 0$$

(3) $t = 0.25, \quad x = 0.75$

$$\frac{W_{3,2} - 2W_{3,1} + \overset{\sin \pi x_3}{W_{3,0}}}{k^2} - \frac{\overset{0\ (BC)}{W_{4,1}} - 2W_{3,1} + W_{2,1}}{h^2} = 0$$

- Improving the Initial Approximation.

$\frac{\partial^2 u}{\partial t^2} \approx$ second-order approx.

$\frac{\partial^2 u}{\partial x^2} \approx$ second-order approx.

$\frac{\partial u}{\partial t}(x, 0) \approx$ first-order approx. (?)

- Centered Difference (2nd-order)

$t = t_1$ (New update from PDEs + IC 2: $g(x)$)

$t = 0$ (IC 1 : $f(x)$)

$t = t_{-1}$

$$\frac{w_{i,1} - w_{i,-1}}{2k} = g(x_i) \quad \cdots \quad (*)$$

a ghost point.

At $t = 0$ and $x = x_i$

ghost

$$\frac{w_{i,1} - 2w_{i,0} + w_{i,-1}}{k^2} - \alpha^2 \frac{w_{i+1,0} - 2w_{i,0} + w_{i-1,0}}{h^2} = 0$$

From $(*)$,

$$w_{i,-1} = w_{i,1} - 2k\,g(x_i)$$

$$\Rightarrow \frac{2w_{i,1} - 2w_{i,0} - 2k\,g(x_i)}{k^2} - \alpha^2 \frac{w_{i+1,0} - 2w_{i,0} + w_{i-1,0}}{h^2} = 0$$