06/02/20

Sec 12.4. An Introduction to the Finite-Element Method

$$-\Delta u = f \quad \text{in} \quad \Omega$$
$$u = 0 \quad \text{on} \quad \partial\Omega$$

$$\left(-\nabla \cdot (K \nabla u) + ru = f \right)$$

- Variational problem

$$I[u] = \frac{1}{2}\int_\Omega [\nabla u]^2 d\vec{x} - \int_\Omega f u \, d\vec{x}$$

Find $u \in V$ that minimizes $I[u]$.

- Function space V

$u \in V \Rightarrow I[u]$ must be finite.

$$\boxed{\int_\Omega [\nabla u]^2 d\vec{x} < \infty}$$

$$\int_\Omega f u \, d\vec{x} \underset{C-S}{\leq} \left(\int_\Omega [f]^2 d\vec{x} \right)^{1/2} \left(\boxed{\int_\Omega [u]^2 d\vec{x}} \right)^{1/2} < \infty$$

$< \infty$

$$V = \{ v : \int_\Omega [\nabla v]^2 d\vec{x} < \infty, \int_\Omega [v]^2 d\vec{x} < \infty, v = 0 \text{ on } \partial\Omega \}$$

$$\left(C_0^2(\Omega) \subsetneq V \right)$$

- Equivalent variational problem

Find $u \in V$ such that

$$\int_\Omega f v \, d\vec{x} = \int_\Omega (-\Delta u) v \, d\vec{x}$$

outward normal vector

$$= \int_\Omega \nabla u \cdot \nabla v \, d\vec{x} - \cancelto{0}{\int_{\partial\Omega} (\nabla u \cdot \vec{n}) v \, d\vec{x}}$$

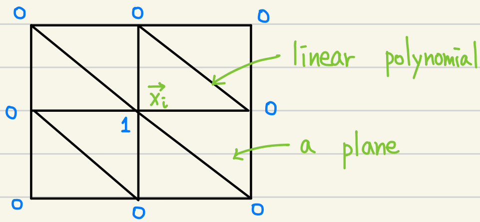
for all $v \in V$.

↳ Bilinear form

$$a(u, v) = \int_\Omega \nabla u \cdot \nabla v \, d\vec{x}$$

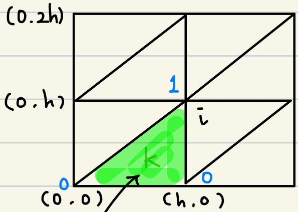
– Finite element space V_h

- Basis functions



$$\phi_i(\vec{x}_i) = 1$$
$$\phi_i(\vec{x}_j) = 0 \quad \text{if} \quad j \neq i$$

ex)



a plane : $z = \phi_i^k(x,y) = ax+by+c$

$\Rightarrow \phi_i^k(h,h) = ah+bh+c = 1$

$\phi_i^k(h,0) = ah+c = 0$

$\phi_i^k(0,0) = c = 0$

$$\Rightarrow \phi_i^k(x,y) = y/h$$

- $v_h \in V_h$

$$v_h(x,y) = \sum_i \alpha_i \phi_i(x,y)$$

$(V_h \subseteq V$ (Why?))

- Finite element method

Find $u_h \in V_h$ such that

$$a(u_h, v_h) = \int_\Omega f v_h \, d\vec{x}$$

for all $v_h \in V_h$. (Galerkin)

↳ ϕ_i for all $1 \le i \le N$

$$\Rightarrow \quad u_h(x,y) = \sum_{j=1}^{N} u_j \phi_j(x,y)$$

$$a(u_h, \phi_i) = \int_\Omega \nabla u_h \cdot \nabla \phi_i \, d\vec{x}$$

$$= \sum_{j=1}^{N} u_j \int_\Omega \nabla \phi_j \cdot \nabla \phi_i \, d\vec{x}$$

$$= \int_\Omega f \phi_i \, d\vec{x}$$

for all $1 \le i \le N$

$\Rightarrow \quad A\vec{x} = \vec{b}$, where

$$(A)_{ij} = \int_\Omega \nabla \phi_j \cdot \nabla \phi_i \, d\vec{x}$$

$$= \int_\Omega \nabla \phi_i \cdot \nabla \phi_j \, d\vec{x} \rightarrow A : \text{symmetric.}$$

$$(\vec{x})_i = u_i$$

$$(\vec{b})_i = \int_\Omega f \phi_i \, d\vec{x}$$