Research Statement

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1 Introduction

My research interest is in the area of deformation quantization. Historically, deformation quantization theory is the mathematical interpretation of the connection between the classical mechanical system and the quantum mechanical system. A deformation quantization problem usually start with a classical system described by a commutative algebra A. By using the information contained in the classical system, the deformation quantization of A is a non commutative algebra B which depends on a parameter $\hbar$ so that for $\hbar = 0$, it becomes A, i.e., $B/\hbar B \cong A$.

One of the greatest milestones in deformation quantization is the famous work of Kontsevich [K] in 1997. It shows the possibility of deformation quantization for Poisson manifolds. Following the idea of Kontevich, one can ask questions about the deformation quantization of coherent sheaves on a Poisson manifold. In full generality, this is a very difficult problem. However, by the paper of [BGP], the first and second order deformation of line bundle can be determined by a certain type of cohomology equation. If we restrict the total space X to be symplectic, a necessary condition for the existence of the quantization deformation of the coherent sheaf given by [Ga] shows that the support of the coherent sheaf must be a coisotropic subvariety of X.

My research goal is to find the criterion of the deformation quantization of coherent sheaves on coisotropic subvarieties. Under the assumption that the line bundle is defined on a Lagrangian subvariety, [BGKP] provides a necessary and sufficient condition of the existence of the deformation quantization. This involves methods such as the lifting of Harish-Chandra torsors and period maps given by [BK]. The research that Prof. Baranovsky and I did in the past is to generalize the result of [BGKP] to the case of vector bundles. Our paper [BC] is published online by the International Mathematics Research Notices on October 9th 2017.

2 Past Research

In work done in collaboration with Baranovsky [BC], we study the sufficient and necessary condition of the deformation quantization of vector bundles on Lagrangian subvarieties.

We consider a smooth Lagrangian subvariety Y in a smooth algebaric variety X. According to an observation of [BK], a choice of deformation quantization $\mathcal{O}_\hbar$ of the structure sheaf $\mathcal{O}_X$ is equivalent to a choice of a lift of the formal coordinate system torsor P_X of X over $\langle \mathrm{Aut}(\mathcal{A}), \mathrm{Der}(\mathcal{A})\rangle$ to a transitive Harish-Chandra torsor $P_\hbar$ over $\langle \mathrm{Aut}(\mathcal{D}), \mathrm{Der}(\mathcal{D})\rangle$, where $\mathcal{A}$ and $\mathcal{D}$ are local structures of $\mathcal{O}_X$ and $\mathcal{O}_\hbar$. Furthermore, after fixing a choice of $\mathcal{O}_\hbar$, the paper [BGKP] shows that a choice of line bundle L on Y and its deformation quantization $L_\hbar$ is equivalent to the choice of a lift of the formal coordinate system torsor $P_\mathcal{J}$ of Y to a transitive Harish-Chandra torsor $P_{\mathcal{D},\mathcal{M}}$ over $\langle \mathrm{Aut}(\mathcal{D},\mathcal{M}), \mathrm{Der}(\mathcal{D},\mathcal{M})\rangle$, where $\mathcal{M}$ is the module structure of the line bundle. By following the idea of [BGKP], we applied the technique of formal geometry to generalize this equivalence relation to the case of vector bundles:

Proposition 1. *A choice of a vector bundle E of rank r on Y and its deformation quantization $E_\hbar$ is equivalent to a choice of a lift of the torsor $\mathcal{P}_\mathcal{J}$ to a transitive Harish-Chandra $\langle \mathrm{Aut}(\mathcal{D},\mathcal{M}_r), \mathrm{Der}(\mathcal{D},\mathcal{M}_r)\rangle$-torsor $\mathcal{P}_{\mathcal{M}_r}$ along the extension of pairs*

$$1 \to \langle GL_\hbar(r), \mathfrak{gl}_\hbar(r)\rangle \to \langle \mathrm{Aut}(\mathcal{D},\mathcal{M}_r), \mathrm{Der}(\mathcal{D},\mathcal{M}_r)\rangle \to \langle \mathrm{Aut}(\mathcal{D})_\mathcal{J}, \mathrm{Der}(\mathcal{D})_\mathcal{J}\rangle \to 1.$$

The criterion of the deformation quantization of vector bundles on Lagrangian subvarieties can therefore be studied by considering the lifting of torsors along a chain of surjections:

$$\begin{aligned}\langle \mathrm{Aut}(\mathcal{D},\mathcal{M}_r), \mathrm{Der}(\mathcal{D},\mathcal{M}_r)\rangle &\twoheadrightarrow \langle \mathrm{Aut}(\mathcal{D},\mathcal{M}_r), \mathrm{Der}(\mathcal{D},\mathcal{M}_r)\rangle/\langle \mathbb{C}^*, \mathbb{C}\rangle \\ \twoheadrightarrow \langle \mathrm{Aut}(\mathcal{D})_\mathcal{J}, \mathrm{Der}(\mathcal{D})_\mathcal{J}\rangle &\times \langle PGL_\hbar(r), \mathfrak{pgl}_\hbar(r)\rangle \twoheadrightarrow \langle \mathrm{Aut}(\mathcal{D})_\mathcal{J}, \mathrm{Der}(\mathcal{D})_\mathcal{J}\rangle\end{aligned}$$

We proved that the groupoid category of lifts of $P_\mathcal{J}$ to a transitive Harish-Chandra torsor P_1 over $\langle \mathrm{Aut}(\mathcal{D})_\mathcal{J}, \mathrm{Der}(\mathcal{D})_\mathcal{J}\rangle \times \langle PGL_\hbar(r), \mathfrak{pgl}_\hbar(r)\rangle$ is equivalent to the category of $PGL_\hbar(r)$-bundles on Y with a flat algebraic connection. Moreover, under the assumption that $\mathcal{O}_\hbar$ has the non-commutative period $[\omega] + h\omega_1 + h^2\omega_2 + \ldots \in H^2_{DR}(X)[[\hbar]]$, for any given P_1, the category of its lifts to a transitive Harish-Chandra torsor P_2 over $\langle \mathrm{Aut}(\mathcal{D},\mathcal{M}_r), \mathrm{Der}(\mathcal{D},\mathcal{M}_r)\rangle/\langle \mathbb{C}^*, \mathbb{C}\rangle$ is equivalent to the category of lifts of the original torsor $P_\mathcal{J}$ to a transitive torsor over $\langle \mathrm{Aut}(\mathcal{D},\mathcal{M}_1)/\mathbb{C}^*, \frac{1}{\hbar}\mathcal{J}/\mathbb{C}\rangle$. The latter is non-empty if and only if in $H^2_{DR}(Y)[[\hbar]]$,

$$j^*(\hbar^2\omega_2 + \hbar^3\omega_3 + \ldots) = 0.$$

Given a choice of the torsor P_2, the groupoid category of its lifts to a transitive Harish-Chandra torsor P over $\langle \mathrm{Aut}(\mathcal{D},\mathcal{M}_r), \mathrm{Der}(\mathcal{D},\mathcal{M}_r)\rangle$ is equivalent to the category of rank r modules E over the Lie algebroid $\mathcal{L}^+(\mathcal{O}_\hbar)$, equipped with an isomorphism of $\mathbb{P}(E)$ and the flat $PGL(r)$-bundle induced from P_1 via the homomorphism

$PGL_\hbar(r) \to PGL(r)$. Such lifts exists if and only if the following equality holds in $H^2_{DR}(Y)$:

$$\frac{1}{r}c_1(E) = \frac{1}{2}c_1(K_Y) + c(\mathcal{L}(\mathcal{O}_\hbar)) = c(\mathcal{L}^+(\mathcal{O}_\hbar))$$

Furthermore, for a given choice of P_1, the groupoid category of its lifts to a transitive Harish-Chandra torsor P over $\langle \mathrm{Aut}(\mathcal{D}, \mathcal{M}_r), \mathrm{Der}(\mathcal{D}, \mathcal{M}_r)\rangle$ - if non-empty - is equivalent to the groupoid category $\mathcal{O}_Y[[\hbar]]^*$-torsors with a flat algebraic connection. More precisely, for any fixed choice P and a flat $\mathcal{O}_Y[[\hbar]]^*$-torsor L there is a well-defined transitive Harish-Chandra torsor $L \star P$, and the functor $L \mapsto L \star P$ gives an equivalence of categories.

All these results can be summarized by the main theorem of our paper [BC]:

Theorem 2. *A rank r vector bundle E on a smooth Lagrangian subvariety $j : Y \to X$ admits a deformation quantization if and only if the following conditions hold:*

1. *$j^*\omega_k = 0$ in $H^2_{DR}(Y)$ for $k \geq 2$;*
2. *the projectivation $\mathbb{P}(E)$ admits a flat algebraic connection;*
3. *the refined first Chern class in $H^2_F(Y)$ satisfies*

$$\frac{1}{r}c_1(E) = \frac{1}{2}c_1(K_Y) + j^*\omega_1;$$

for the canonical lift of $j^*\omega_1$ to $H^2_F(Y)$ representing the class of the Picard algebroid $\mathcal{L}(\mathcal{O}_\hbar) = \mathcal{T}or_1^{\mathcal{O}_\hbar}(\mathcal{O}_Y, \mathcal{O}_Y)$.

If nonempty, the set of equivalence classes of all rank r deformation quantizations on Y for various E has a free action of the group $\mathcal{G}$ of isomorphism classes of $\mathcal{O}_Y[[\hbar]]^*$-torsors with a flat algebraic connection. The set of orbits for this action may be identified with the space of all $PGL(r, \mathbb{C}[[\hbar]])$ bundles with a flat algebraic connection.

3 Recent Research

In the case of coisotropic subvarieties, [BGP] provides some sufficient conditions stating when a line bundle admits a first order deformation quantization. By assuming some certain obstruction classes trivial, they also provide a necessary condition for the second order deformation quantizations.

My approach on the infinite order deformation quantization of line bundle on coisotropic subvarieties is based on the technique of formal geometry and L_∞-algebras. As in the case of Lagrangian subvarieties, the deforamtion quantization of line bundle on coisotropic subvarieties is also equivalence to the lifting of a torsor along a Harish-Chandra extension. However, the kernel of the Lie algebra part of this extension is no longer abelian. For this reason, I found it is necensary to view the lifting of the torsor as the curved Maurer-Cartan solution of an L_∞-algebra of Cech complexes associated to a non-ablian group.

During the last few months, I made several calculations on the L_∞ structure relating to the lifting of torsors. The criterion on the solution of curved Maurer-Cartan equation begin to become clear after constructing the L_∞ morphisms between the L_∞ structure of the Lie algebra sheaf and the de Rham cohomology of the null foliation. The expectation is that the equation of cohomology classes will be exactly the same, but the de Rham complex should be on the coisotropic subvariety and along coordinates tangent to null-foliation.

4 Future Research

The holy grail of deforamtion quatization is to find the criterion of the deformation quantization of coherent sheaf on coisotropic subvarieties. In theory, our methods on Lagrangian subvarties should work for vector bundles on smooth coisotropic subvarieties. In this case, the conormal bundle embeds as a null-foliation subbundle of the tangent bundle. Consequently, the full de Rham complex will be replaced by the normal de Rham complex.

Another valuable question that I would like to answer in the future is that if a pair of vector bundles E_1, E_2 are deformed to $\mathcal{E}_1, \mathcal{E}_2$, how to desceribe $\mathrm{Hom}_{\mathcal{O}_\hbar}(\mathcal{E}_1, \mathcal{E}_2)$ in terms of E_1, E_2. It would be interesting to compare the de Rham cohomology of the local system $\mathrm{Hom}_{\mathcal{O}_Y}(E_1, E_2)$ and the groups $\mathrm{Ext}_{\mathcal{O}_\hbar}(\mathcal{E}_1, \mathcal{E}_2)$ for deformation quantization of E_1, E_2, respectively. I expect $\mathrm{Ext}^i_{\mathcal{O}_\hbar}(\mathcal{E}_1, \mathcal{E}_2)$ to be related to Donaldson-Thomas invariants.

References

[B] Brylinski, J.-L.: Loop spaces, characteristic classes and geometric quantization. Reprint of the 1993 edition. *Modern Birkhäuser Classics. Birkhäuser Boston, Inc.,* Boston, MA, 2008.

[BB] Beilinson, A.; Bernstein, J.: A proof of Jantzen Conjectures. *Advances in Soviet Mathematics*, Vol. 16, part 1 (1993), pp. 1–50.

[BC] Baranovsky, V.; Chen, T. :Quantization of vector bundles on Lagrangian subvarieties. *International Mathematics Research Notices*, rnx230.

[BGKP] Baranovsky, V.; Ginzburg, V.; Kaledin, D.; Pecharich, J.: Quantization of line bundles on lagrangian subvarieties. *Selecta Math. (N.S.)* **22** (2016), no. 1, 1–25.

[BGP] Baranovsky, V.; Ginzburg, V.; Pecharich, J.: Deformation of Line Bundles on Coisotropic Subvarieties. *The Quarterly Journal of Mathematics* (2012), 525–537.

[BK] Bezrukavnikov, R.; Kaledin, D.: Fedosov quantization in algebraic context. *Mosc. Math. J.* **4** (2004), no. 3, 559–592, 782.

[Ga] Gabber, O.: The integrability of the characteristic variety. *Amer. J. Math.* **103** (1981), 445-468.

[K] Kontsevich, M.:Deformation quantization of Poisson manifolds, *Lett. Math. Phys.* **66** (2003), 157-216.

Mosc. Math. J. **4** (2004), no. 3, 559–592, 782.

[P] Pecharich, J.: Deformations of vector bundles on coisotropic subvarieties via the Atiyah class, preprint arxiv 1010.3671.