Research Statement

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1 Introduction

My research interest is in the area of deformation quantization. Historically, deformation quantization theory is the mathematical interpretation of the connection between the classical mechanical system and the quantum mechanical system. A deformation quantization problem usually start with a classical system described by a commutative algebra A. By using the information contained in the classical system, the deformation quantization of A is to turn it into a non-commutative algebra B which depends on a parameter $\hbar$. So that for $\hbar = 0$, it becomes A, i.e., $B/\hbar B \cong A$.

For a more concrete example, assume that X is a symplectic manifold. Roughly speaking, the structure sheaf $\mathcal{O}_X$ describes the correspondence between any open subset U of X and the ring of smooth functions $\mathcal{O}(U)$ defined on U. In a deformation quantization $\mathcal{O}_\hbar$ of $\mathcal{O}_X$, $\mathcal{O}(U)$ is replaced by a non-commutative algebra $\mathcal{O}_\hbar(U)$ over the formal power series $\mathbb{C}[[\hbar]]$ such that $\mathcal{O}_\hbar(U)/\hbar\mathcal{O}_\hbar(U) \cong \mathcal{O}(U)$. As vector spaces, $\mathcal{O}_\hbar(U)$ is isomorphic to $\mathcal{O}(U)[[\hbar]]$. This allows us to write the product in the non-commutative algebra in the form of

$$f * g = fg + \hbar P(df, dg) + \hbar^2 \cdots$$

where $P = \sum a_{ij} \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial y_j}$ comes from the symplectic form ω.

Suppose Y to be a submanifold of X. E is the vector bundle on Y. Since a vector bundle is a geometric object which allow us to generalize vector valued functions on manifold, for any open subset U of Y, $E(U)$ consists of sections of E over U. Furthermore, $E(U)$ is a module over $\mathcal{O}(U)$. My research goal is to find the criterion of deformation quantization $E_\hbar$ of E over $\mathcal{O}_\hbar$.

In full generality, finding the criterion of deformation quantization of a coherent sheaf is a very difficult problem. In [Ga], Gabber shows that a necessary condition for the existence of the quantization deformation of vector bundle is that the submanifold Y must be coisotropic. Under the assumption that the line bundle is defined on a Lagrangian subvariety on X, [BGKP] provides a necessary and sufficient condition of the existence of the deformation quantization. This involves methods such as the lifting of Harish-Chandra torsors and period maps given by [BK]. The research that Prof. Baranovsky and I did in the past is to generalize the result of [BGKP] to the case of vector bundles. Our paper [BC] is published online by the International Mathematics Research Notices on October 9th 2017.

2 Past Research

In work done in collaboration with Baranovsky [BC], we study the sufficient and necessary condition of the deformation quantization of vector bundles on Lagrangian subvarieties.

We consider a smooth Lagrangian submanifold Y of a symplectic manifold X. In quantum mechanics, X can be viewed as the phase space which has coordinates $p_1, \cdots, p_n, q_1, \cdots, q_n$. The lagrangian submanifold Y is the subspace of X containing p_i coordinates only. According to an observation of [BK], a choice of deformation quantization $\mathcal{O}_\hbar$ of the structure sheaf $\mathcal{O}_X$ is equivalent to a choice of a lift of the principal bundle of group of automorphisms of X with Lie algebra actions to the principal bundle over the group of automorphisms of deformed structure of X.

$$\left\{ \begin{array}{c} \text{Deformation quantizations} \\ \text{of the structure sheaf } \mathcal{O}_X \end{array} \right\} \overset{\text{1-1}}{\longleftrightarrow} \left\{ \begin{array}{c} \text{Lifts of the principal bundle} \\ \text{of group of automorphisms of } X \end{array} \right\}$$

The different choices of $\mathcal{O}_\hbar$ can be classified by

$$[\omega] + \hbar\omega_1 + \hbar^2\omega_2 + \cdots \in H^2_{\mathrm{DR}}(X)[[\hbar]].$$

After fixing a choice of $\mathcal{O}_\hbar$ and a vector bundle E on Y, the criterion of deformation quantization of E can be summarized by the main theorem of our paper [BC]:

Theorem 1. *A rank r vector bundle E on a smooth Lagrangian subvariety $j : Y \to X$ admits a deformation quantization if and only if the following conditions hold:*

1. $\omega_k|_Y = 0$ *in* $H^2_{\mathrm{DR}}(Y)$ *for* $k \geq 2$*;*
2. *the projectivation* $\mathbb{P}(E)$ *admits a flat algebraic connection;*
3. *the refined first Chern class in* $H^2_{\mathrm{DR}}(Y)$ *satisfies*
$$\frac{1}{r}c_1(E) = \frac{1}{2}c_1(K_Y) + \omega_1|_Y;$$

If nonempty, the set of equivalence classes of all rank r deformation quantizations on Y for various E has a free action of the group $\mathcal{G}$ of isomorphism classes of $\mathcal{O}_Y[[\hbar]]^*$-bundles with a flat algebraic connection. The set of orbits for this action may be identified with the space of all $PGL(r, \mathbb{C}[[\hbar]])$ bundles with a flat algebraic connection.

To prove the main theorem, we applied the technique of formal geometry to genearlize the equivalence relation to the case of vector bundles. A choice of the deformation quantization of a vector bundle E on Y is equivalent to the choice of a lift of the principal bundle of group of automorphisms of Y with Lie algebra actions to the principal bundle over the group of automorphisms of deformed module structure of E.

$$\left\{ \begin{array}{c} \text{Deformation quantizations} \\ \text{of the vector bundle } E \text{ on } Y \end{array} \right\} \overset{\text{1-1}}{\longleftrightarrow} \left\{ \begin{array}{c} \text{Lifts of the principal bundle} \\ \text{w.r.t } Y \text{ to the one over} \\ \text{the group of automorphisms of} \\ \text{deformed module structure of } E \end{array} \right\}$$

More precisely, the principle bundle that we used to characterize deformation quantizations is called Harish-Chandra torsor. The group of automorphisms and the Lie algebra of the derivation of the local structures form Harish-Chandra pairs. Deformation quantizations of vector bundles can therefore be described by lifting the Harish-Chandra torsor along the extension of Harish-Chandra pairs.

After calculating the conditions on lifting the torsors, we found that it is necessary and sufficient to require that all $\omega_i|_Y, i > 1$ to be equal to zero in de Rham cohomology, the projectization of the vector bundle should have flat connection, and the first Chern class should satisfied a certain equation.

3 Recent Research

In the case of coisotropic subvarieties, [BGP] provides some sufficient conditions stating when a line bundle admits a first order deformation quantization. By assuming some certain obstruction classes trivial, they also provide a necessary condition for the second order deformation quantizations.

My approach on the infinite order deformation quantization of line bundle on coisotropic subvarieties is based on the technique of formal geometry and L_∞-algebras. As in the case of Lagrangian subvarieties, the deforamtion quantization of line bundle on coisotropic subvarieties is also equivalence to the lifting of a torsor along a Harish-Chandra extension. However, the kernel of the Lie algebra part of this extension is no longer abelian. For this reason, I found it is necensary to view the lifting of the torsor as the curved Maurer-Cartan solution of an L_∞-algebra of Cech complexes associated to a non-ablian group.

During the last few months, I made several calculations on the L_∞ structure relating to the lifting of torsors. The criterion on the solution of curved Maurer-Cartan equation begin to become clear after constructing the L_∞ morphisms between the L_∞ structure of the Lie algebra sheaf and the de Rham cohomology of the null foliation. The expectation is that the equation of cohomology classes will be exactly the same, but the de Rham complex should be on the coisotropic subvariety and along coordinates tangent to null-foliation.

4 Future Research

The holy grail of deforamtion quatization is to find the criterion of the deformation quantization of coherent sheaf on coisotropic subvarieties. In theory, our methods on Lagrangian subvarties should work for vector bundles on smooth coisotropic subvarieties. In this case, the conormal bundle embeds as a null-foliation subbundle of the tangent bundle. Consequently, the full de Rham complex will be replaced by the normal de Rham complex.

Another valuable question that I would like to answer in the future is that if a pair of vector bundles E_1, E_2 are deformed to $\mathcal{E}_1, \mathcal{E}_2$, how to desceribe $\mathrm{Hom}_{\mathcal{O}_\hbar}(\mathcal{E}_1, \mathcal{E}_2)$ in terms of E_1, E_2. It would be interesting to compare the de Rham cohomology of the local system $\mathrm{Hom}_{\mathcal{O}_Y}(E_1, E_2)$ and the groups $\mathrm{Ext}_{\mathcal{O}_\hbar}(\mathcal{E}_1, \mathcal{E}_2)$ for deformation quantization of E_1, E_2, respectively. I expect $\mathrm{Ext}^i_{\mathcal{O}_\hbar}(\mathcal{E}_1, \mathcal{E}_2)$ to be related to Donaldson-Thomas invariants.

A more technical version of my research statement can be found on my personal website http://sites.uci.edu/taijic.

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